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LETTER TO THE EDITOR

Symmetry group of partial differential equations and of differential difference equations: the Toda lattice versus the Korteweg–de Vries equation

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Abstract. In this work we correlate the symmetry group of the continuous transformations of the Toda lattice to that of the Korteweg–de Vries equation. We show how, by taking into account the continuous limit of the Toda, the four-parameter symmetry group of the Toda is contained in that of the KdV equation. By an inverse process, discretization of the symmetry group of the KdV, we find a discrete element of the symmetry group of the Toda lattice, which gives, by symmetry reduction, its soliton solution.

1. Introduction

In a recent article [1] the construction of the symmetry group for differential difference equations was introduced and applied, as an example, to the Toda lattice equation:

$$u_{\tau\tau}(n) = e^{u(n-1)-u(n)} - e^{u(n)-u(n+1)}. \tag{1}$$

This equation is the prototype of the nonlinear differential difference equations (DDE) which are integrable via the spectral transform, possess an infinity of conservation laws and of higher symmetries, can be written in Hamiltonian form, etc. In the case of nonlinear partial differential equations (PDE) the best known integrable equation is the Korteweg–de Vries equation (KdV):

$$q_t = q_{xxx} + 6qq_x. \tag{2}$$

The two equations (1), (2) are related as there exists a continuous limit, i.e. when the lattice spacing Δ goes to zero, which reduces the Toda lattice to the potential KdV equation, the KdV written in terms of the potential $v(x,t) = \int^x q(x',t)\,dx'$:

$$v_{xt} = v_{xxxx} + 6v_xv_{xx}. \tag{3}$$

In fact, by defining

$$u(n,\tau) = -\tfrac{1}{2}\Delta v(x,t) \qquad x = (n-\tau)\Delta \qquad t = -\tfrac{1}{24}\Delta^3\tau \tag{4}$$

equation (1) is reduced, by carrying out a Taylor expansion around the point x, to (3) up to terms of order Δ^2. In the limiting process we expand the exponential factor and this gives the nonlinear term appearing in the potential KdV equation (3).

Taking into account this result, a few questions come immediately to mind: which is the relation between the corresponding symmetry groups? Can we gain some information about the symmetry group of one equation by knowledge of that of the other? The content of this letter is devoted to answering, at least partially, these questions.

To do so, and to fix the notation, here we report the symmetry vector of (1) and (3) as one can find in the literature or obtain by standard techniques [2]; in the case of the Toda lattice equation (1) we have

$$\hat{v} = (\alpha_1 + \alpha_2 \tau)\partial_\tau + (2n\alpha_2 + \alpha_3 \tau + \alpha_4)\partial_{u(n,\tau)} \tag{5}$$

where $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ are constant parameters, while for the potential KdV (3) we have

$$\hat{w} = (\beta_1 + \beta_2 t + \tfrac{1}{3}\beta_3 x)\partial_x + (\beta_4 + \beta_3 t)\partial_t + (-\tfrac{1}{3}\beta_3 v - \tfrac{1}{6}\beta_2 x + \gamma(t))\partial_v \tag{6}$$

where $\beta_1, \beta_2, \beta_3, \beta_4$ are constant parameters and γ is an arbitrary function of t.

In correspondence with each one-parameter symmetry vector of (5) and (6) we can construct a one-parameter group of transformations by exponentiating the vector field. We have

$$V_{\alpha_1}: \begin{cases} \tau' = \tau + \lambda_1 \\ n' = n \\ u'(n', \tau') = u(n, \tau) \end{cases} \qquad V_{\alpha_2}: \begin{cases} \tau' = \tau\, e^{\lambda_2} \\ n' = n \\ u'(n', \tau') = u(n, \tau) + 2n\lambda_2 \end{cases}$$

$$V_{\alpha_3}: \begin{cases} \tau' = \tau \\ n' = n \\ u'(n', \tau') = u(n, \tau) + \tau\lambda_3 \end{cases} \qquad V_{\alpha_4}: \begin{cases} \tau' = \tau \\ n' = n \\ u'(n', \tau') = u(n, \tau) + \lambda_4 \end{cases} \tag{7}$$

$$W_{\beta_1}: \begin{cases} x' = x + \mu_1 \\ t' = t \\ v'(x', t') = v(x, t) \end{cases} \qquad W_{\beta_2}: \begin{cases} x' = x + t\mu_2 \\ t' = t \\ v'(x', t') = v(x, t) - \tfrac{1}{6}x\mu_2 - \tfrac{1}{12}t\mu_2^2 \end{cases}$$

$$W_{\beta_3}: \begin{cases} x' = x\, e^{\mu_3/3} \\ = t' = t\, e^{\mu_3} \\ = v'(x', t') = v(x, t)\, e^{-\mu_3/3} \end{cases} \qquad W_{\beta_4}: \begin{cases} x' = x \\ t' = t + \mu_4 \\ v'(x', t') = v(x, t) \end{cases} \tag{8}$$

$$W_{\beta_5}: \begin{cases} x' = x \\ t' = t \\ v'(x', t') = v(x, t) + \gamma(t) \end{cases}$$

where $\{\lambda_i\}_{i=1}^4$, $\{\mu_i\}_{i=1}^5$ are two sets of real group parameters and by, say V_{α_1}, we mean the group transformation obtained by choosing in the vector field $\hat{v}$ only the parameter α_1 different from zero, which, with no restriction, we can set equal to 1.

Next we consider the one-parameter transformations for the potential KdV equation which one obtains starting from those of the Toda lattice (7), and then look for discrete transformations of the Toda obtained as discretization of the one-parameter transformations (8). There follows a discussion of the results obtained and of future perspectives in this field.

We now perform the continuous limit for $\Delta \to 0$ of the one-parameter transformations (7) of the Toda lattice. As the corresponding limit at the level of the equation is well defined and gives rise to the potential KdV equation, we can say that the resulting one-parameter transformations will leave the potential KdV invariant. As we shall see, in effect, the resulting one-parameter transformations are a subclass of the possible transformations of the potential KdV.

Let us start from V_{α_1} in equation (7) and take into account the definitions (4) which we will rewrite in the form

$$u(n,\tau)=-\frac{\Delta}{2}v(x,t) \qquad n=-\frac{24t}{\Delta^3}+\frac{x}{\Delta} \qquad \tau=-\frac{24}{\Delta^3}t. \tag{9}$$

Introducing (9) into V_{α_1}, we get

$$\begin{aligned} &t'=t-\Delta^3\lambda_1/24 \qquad x'=x-\Delta\lambda_1 \\ &v'(x',t')=v(x,t). \end{aligned} \tag{10}$$

We can now choose λ_1 as a function of Δ in such a way as to get a finite result. If λ_1 is constant then when $\Delta\to 0$ (10) will give just the identity transformation but, by choosing $\Delta\lambda_1$ to be constant, say μ_1, then (10) provides the transformation W_{β_1} for the potential KdV. The choice $\Delta^2\lambda_1$ constant is not admissible as, when $\Delta\to 0$, (10) will give rise to diverging x'.

We now consider together V_{α_2} and V_{α_3}; in such cases we get

$$\begin{aligned} &t'=t\,\mathrm{e}^{\lambda_2} \qquad x'=x+24t(\mathrm{e}^{\lambda_2}-1)/\Delta^2 \\ &v'(x',t')=v(x,t)+4\lambda_2(24t-\Delta^2x)/\Delta^4 \end{aligned} \tag{11}$$

and

$$\begin{aligned} &t'=t \qquad x'=x \\ &v'(x',t')=v(x,t)+48\lambda_3 t/\Delta^4. \end{aligned} \tag{12}$$

Let us analyse (11). λ_2 constant is not admissible as diverging terms will appear, so a first natural choice is to set $24(\mathrm{e}^{\lambda_2}-1)/\Delta^2$ equal to a constant, say μ_2. In such a case (11) would become

$$\begin{aligned} &t'=t \qquad x'=x+\mu_2 t \\ &v'(x',t')=v(x,t)-\tfrac{1}{6}\mu_2 x+4\mu_2 t/\Delta^2-\tfrac{1}{12}\mu_2^2 t. \end{aligned} \tag{13}$$

The transformation (13) looks like W_{β_2} but as $\Delta\to 0$ it still contains a divergent term. However (12) provides us with a diverging term of the same form as that of (13); then, by choosing $\lambda_3=-\mu_2\Delta^2/12$ and combining transformations (12) and (13) we get W_{β_2}. By the choice $96\lambda_2/\Delta^4$ and $48\lambda_3/\Delta^4$ constant, say μ_5, both (11) and (12) reduce to W_{β_5} with $\gamma(t)=t$.

In the same manner we can analyse V_{α_4} and by choosing $-2\lambda_4/\Delta$ constant, say μ_5, we get again W_{β_5}, however, this time, with $\gamma(t)$ constant.

It is worthwhile to notice here that the three transformations, W_{β_1}, W_{β_2} and W_{β_5}, we have obtained in this way, form a subgroup of the whole group of transformations of the potential KdV.

In principle, to recover the shift operator from a differential operator we need to consider an infinite series of terms. This would imply that the transformation (4) is not sufficient to define uniquely the Toda lattice starting from the potential KdV. Naturally the relation between $(v(x,t),x,t)$ and $(u(n,\tau),n,\tau)$ given by (4) is well defined and can be applied to pass from one-parameter transformations of the potential KdV to those of a DDE, which *a priori* may not be the Toda lattice. Even so, we consider it worthwhile to carry out these calculations because, when they give results valid for the Toda lattice, these are very interesting as they are discrete transformations, objects which cannot be obtained by any infinitesimal technique.

Let us consider W_{β_1} and apply to it the transformation (4); we get

$$n' = n + \mu_1/\Delta \qquad \tau' = \tau$$
$$u'(n', \tau') = u(n, \tau). \tag{14}$$

To obtain a consistent transformation we must require that μ_1/Δ be a constant and in particular, an integer number. In such a way (14) is a discrete transformation for the Toda lattice whose existence can be proved by direct computation.

W_{β_2} gives

$$n' = n - \tfrac{1}{24}\Delta^2\mu_2\tau \qquad \tau' = \tau$$
$$u'(n', \tau') = u(n, \tau) + \tfrac{1}{12}\Delta^2\mu_2 n - \tfrac{1}{12}\Delta^2\mu_2\tau(1 + \tfrac{1}{48}\Delta^2\mu_2). \tag{15}$$

As τ is a continuous variable the only possible choices of μ_2, such that (15) is coherent, are such that $\mu_2\Delta^2 \to 0$; if we choose μ_2 constant, then (15) provides the identity transformation.

Also in the case of W_{β_3} the only consistent transformation is the identity transformation, while W_{β_4} gives

$$n' = n - 24\mu_4/\Delta^3 \qquad \tau' = \tau - 24\mu_4/\Delta^3$$
$$u'(n', \tau') = u(n, \tau). \tag{16}$$

This transformation gives a proper result as long as we choose $-24\mu_4/\Delta^3$ integer, say λ_4. Combining (16) with (14) with a proper choice of the group parameters involved we can obtain from them V_{α_1}.

W_{β_5} gives

$$n' = n \qquad \tau' = \tau$$
$$u'(n', \tau') = u(n, \tau) - \tfrac{1}{2}\Delta\mu_5\gamma(-\tfrac{1}{24}\tau\Delta^3). \tag{17}$$

The resulting transformation depends on the arbitrary function $\gamma(-\frac{1}{24}\Delta^3\tau)$. Due to the arbitrariness of the function γ we can write it as $u' = u + \lambda_5 g(\tau)$. However the only possible transformations of this form are V_{α_4} and V_{α_3} corresponding to g constant or linear in τ.

As we said above, by a limiting procedure, the continuous limit of the transformations of the Toda lattice provides a subgroup of the whole group of point transformations of the potential KdV. This is due to the fact that in general a discrete equation has a continuous symmetry group of lower dimension than that of the corresponding partial differential equation. This is even more clear in the case of the equation:

$$u_\tau = u(n)\{[u(n-1)u(n) + u(n-1) + u(n-2) - 6] - [u(n+1)u(n) + u(n+1) + u(n+2) - 6]\} \tag{18}$$

which is an integrable nonlinear DDE of the same hierarchy as the Toda lattice itself [3]. Even if completely integrable, (18) has only a continuous symmetry (i.e. is only invariant under τ-translations and its continuous limit) the KdV equation (2), is obtained as $\Delta \to 0$ with the transformation

$$u(n, \tau) = 1 + \Delta^2 q(x, t)$$
$$x = n\Delta \qquad t = -2\Delta^3\tau. \tag{19}$$

Equation (18) has in itself polynomial nonlinearities and thus the transformation (19) gives, when introduced into the KdV, a better approximation to (18). Even so when we transform the symmetry group of the KdV according to (19) some of the transformations give rise to wrong discrete point transformation for (18) while the continuous part is correctly recovered.

The inverse transformation allows us to a discrete transformation for the Toda lattice. This transformation could have been obtained by a direct analysis of the Toda equation. Taking into account this discrete transformation $[n'=n+m, \tau'=\tau, u'(n',\tau')=u(n,\tau)]$ and the τ-translation V_{α_1}, we can, setting $\lambda_1=\alpha m/\sinh(\alpha)$ with α an arbitrary parameter, obtain that the variable $\xi=n(\alpha/\sinh\alpha)-\tau$ is a symmetry variable for our equation. By reducing the Toda lattice with respect to it we obtain:

$$u_{\xi\xi}=\exp\{u(\xi-\alpha/\sinh\alpha)-u(\xi)\}-\exp\{u(\xi)-u(\xi+\alpha/\sinh\alpha)\} \qquad (20)$$

whose solution is the well known soliton solution of the Toda lattice [4]

$$u(n,\tau)=u(\xi)=\ln\frac{1+\exp[2(\sinh(\alpha)\xi-\alpha)]}{1+\exp[2\sinh(\alpha)\xi]}. \qquad (21)$$

So, concluding, the analysis of the relation between continuous point transformations of PDE and DDE allowed us to get, from the PDE discrete symmetries of the DDE. However in this process one also gets transformations which do not leave the DDE invariant. This opens up the following questions:

1. Are there some ways of discretizing the PDE which project the group of its point transformations into that of the DDE?

2. Are we able in this way to get all discrete symmetries of the DDE?

Work on answering these questions is in progress.

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